TOWARDS CULTURAL ANALYSIS OF CONTENT: PROBLEMS WITH VARIATION IN PRIMARY SCHOOL[1]

Maria G. Bartolini Bussi
Rita Canalini and Franca Ferri

Dipartimento di Matematica – Università di Modena e Reggio Emilia (Italy)

bartolini@unimore.it

In memory of Filippo Spagnolo

1. INTRODUCTION

Mathematical knowledge needed for teaching is now in the foreground in international Conferences (e. g. Bartolini Bussi et al. 2011). The theoretical construct of *Cultural Analysis of Content* (CAC) has been introduced in mathematics teacher education by Boero & Guala (2008) as follows: "presenting mathematics as an evolving discipline, with different levels of rigor both at a specific moment in history (according to the cultural environment and specific needs), and across history, and as a domain of culture as a set of interrelated cultural tools and social practices, which can be inherited over generations (p. 223)". According to these authors, "CAC can lead teachers to radically question their beliefs concerning mathematics in general and specific subject matter in particular (p. 223)". Boero & Guala (2008) highlight the difficulty of finding suitable tasks for teacher education in well established school mathematics area. Bartolini Bussi (2011) has analyzed, according to CAC, the topic of place value representation of whole numbers in workshops for pre-primary and primary school teachers. In this paper, a further example is discussed concerning the area of *word problems in primary school*. Teachers' questioning on their own beliefs is prompted by the intercultural analysis of a meaningful example hinting at the Chinese teaching method of *problems with variation*. Some intercultural workshops for primary school teachers (each running for around 12-16 hours and involving 20-30 participants) have been held since 2007 (in-service and pre-service education). In the workshops, the method of problems with variation is contextualized, with also homeworks and additional readings, within Chinese culture, with some "rudiments" about written language and numerals, system of values, curricula (Spagnolo and Di Paola, 2010), classroom organization, traditional artefacts and textbooks, because outlandish and isolated teaching strategies could not substantially influence teachers' beliefs and general school practice. Problems with variation have been introduced by means of the careful analysis of a paradigmatic case of a system of *nine problems* (see the table 1), taken from the Chinese textbook "Mathematics" for the second semester of the second grade (Shu Xue, 1996).

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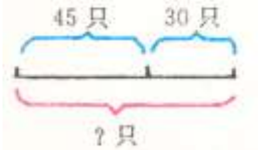
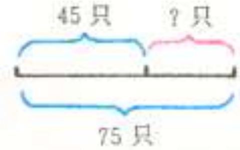
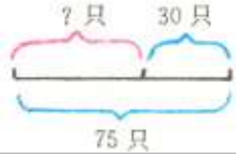
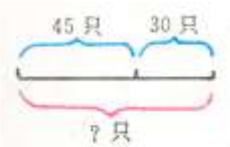
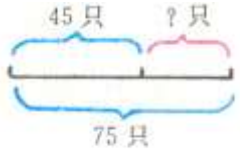
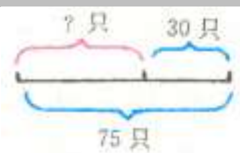
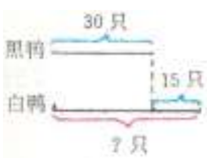
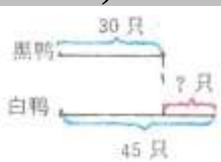
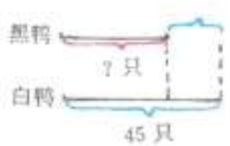
First solve the nine problems below. Then explain why they have been arranged in rows and columns in this way, finding relationships		
(1)**In the river there are** 45 white ducks and 30 black ducks. All together how many ducks are there?	(2)**In the river there are** white ducks and black ducks. All together there are 75 ducks. 45 are white ducks. How many black ducks are there?	(3)**In the river there are** white ducks and black ducks. All together there are 75 ducks. 30 are black ducks. How many white ducks are there?
		
(1) **In the river there is** a group of ducks. 30 ducks swim away. 45 ducks are still there. How many ducks are in the group (at the beginning)?	(2) **In the river there are** 75 ducks. Some ducks swim away. There are still 45 ducks. How many ducks have swum away?	(3) **In the river there are** 75 ducks. 30 ducks swim away. How many ducks are still there?
		
(1) **In the river there are** 30 black ducks. White ducks are 15 more than black ducks (black ducks are 15 less than white ducks). How many white ducks are there?	(2) **In the river there are** 30 black ducks and 45 white ducks. How many white ducks more than black ducks (How many black ducks less than white ducks)?	(3) **In the river there are** 45 white ducks. Black ducks are 15 less than white ducks (white ducks are 15 more than black ducks). How many black ducks are there?
		

Table 1: The nine problems of ducks (Shu Xue, 1996)[2]

In the following, the theoretical framework of the whole program of mathematics teacher education is sketched together with a very short review of the literature about the Chinese method of problems with variation. Then two exemplary teaching experiments, concerning additive and multiplicative problems, inspired by the Chinese textbooks, are discussed: they have been developed by two primary school teachers (Rita Canalini and Franca Ferri), who, after taking part in the early workshops, have exploited this method in their

[2] translated by Patrizia Bartolini.

own classrooms. The discussion shows, on the one hand, how the CAC has modified their approach to word problems, opening new avenues for research, and, on the other hand, how slow is, in general, the process of modifying teachers' own beliefs and to change classroom practice.

2. THEORETICAL FRAMEWORK

A) Semiotic mediation: the didactical cycle.

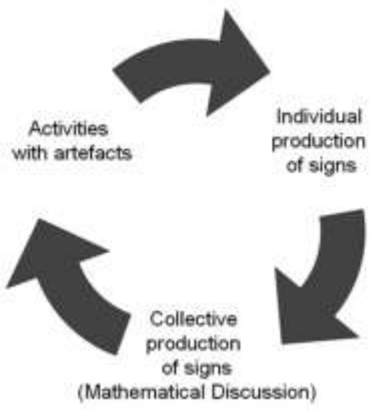


Figure 1:
The didactical cycle

The program for teacher education chaired by Bartolini Bussi draws on the theoretical framework of *semiotic mediation* after a Vygotskian approach. The framework has been elaborated by Bartolini Bussi and Mariotti (2008) exploiting an extensive corpus of data collected in dozens of teaching experiments, carried out at all school level (from pre-primary to secondary) in cooperation with teacher-researchers. The framework has been originally developed to cover the experimental activities with some kind of artefacts, from manipulatives (e.g. abacus, compass) to softwares (e. g. Cabri), and has recently been extended to artefacts as texts and drawings (Mariotti & Maracci, in press). This paper belongs to this second trend. In this paper, because of space limits, we mention only the didactical cycle that allows the organization of classroom activity as a long term process and avoid the discussion (although very interesting) of semiotic mediation. The fig. 1 shows the articulation of classroom processes over time, from individual or small group activity prompted by tasks to be solved by means of suitable artefacts, to the production of solutions by individuals up to the collective production of shared signs (e.g. texts with drawings or schemes) in *mathematical discussion* orchestrated by the teacher. Also *written dialogue* (Bartolini Bussi & Mariotti, 2008, p. 763) is a didactical technique very useful in this context: during the individual task the teacher passes between pupils' desks and interacts with each pupil writing a short comment on the worksheet to ask for explanation, to re-launch the process and the like.

B) Problems with variation.

Problems with variation (变式 *biànshì*) refer to a traditional method used in Chinese mathematics education. From an *example problem* other problems are derived (*biànshì problems*) where some features are changed. In this way, students are encouraged to extract the essential elements of the problem-solving method of the example and adapt them to other problems (Gu, Huang and Marton, 2004; Sun, Wong and Lam, 2005; Cai and Nie, 2007). In the table 1 there are nine problems about ducks. In each row, the example problem is shaded; the others are *biànshì* problems. In the Chinese tradition, where examinations play a very important social role, *biànshì* problems serve as the bridge between textbooks problems and examination problems. They meet also the need of teaching large size classrooms (up to 50-60 students) in China, offering different levels of difficulty. Moreover *biànshì* problems pave the way towards the construction of general algorithms for problem solving. In the table 1 these algorithms are visually represented by the nine coloured schemes, where the unknown and data are represented by red and blue braces.

3. PROBLEMS WITH VARIATION AS A PROMPT FOR THE CULTURAL ANALYSIS OF CONTENT.

The set of the nine problems on ducks is a good prompt to create the "surprise" effect needed to foster changes in beliefs in the CAC (Boero & Guala, 2008): the number size; the joint presentation of addition and subtraction; the very dry and concise texts which mirror the Chinese style; the great deal of problems; the presence of graphic schemes; the high level of the request "to explain the arrangement". In the Chinese textbook no indication is given about the management of this situation in the classroom, hence Franca Ferri and Rita Canalini, two experienced teacher-researchers, take up the challenge, interpreting it within the construct of didactical cycle (see above), and tested the problems in their own classrooms. Short reports of their experiments follow.

4. EXAMPLE 1: ADDITIVE PROBLEMS IN THE SECOND GRADE (Franca Ferri)

Context. The classroom has 22 pupils, taught to cope with complex situations since the beginning of the 1st grade (with the same teacher). They like mathematics, are glad to accept challenges and consider important to discuss about mathematics. They are new to graphic schemes in problem solving. In the classroom there are also low achievers (e.g. Giacomo is a boy with learning disorders) and a high achiever (Arianna). The others are medium level pupils.

The experiment consists in 4 main tasks (to be solved either individually or in pairs-trios) with interlaced discussions orchestrated by the teacher (see the didactical cycle in the fig. 1 above). The teacher chooses to start from the table of the nine problems and to maintain the dry and concise structure of the Chinese text, but *cuts the graphic schemes*, coming back to them only in the fourth task.

4.1. The first task (January 27, 2010).

Task. The table of the fig. 1 (in Italian) is given to each pupil, with the following task: *Observe, reflect and solve (Chinese problems).*

Results. 18 pupils are present. 9 pupils solve all the 9 problems. 5 pupils make some mistake. 4 pupils do not solve the problems in the third row.

Mathematical Discussion. Immediately after the individual solution, a mathematical discussion is orchestrated by the teacher. The prompt follows.

Teacher*: You have solved well or badly the problems I have given. I have told you that there are Chinese, as they have been translated from a Chinese textbook for the second grade. I have given you nine problems and you might have thought that I went mad ...*[she smiles]
Lorenzo [with emphasis]*: We have never solved so many problems all together.*
Teacher*: I have given them all together, because they were together in the Chinese textbook and also because I have thought that they were a bit special and could stay together.*
Mohassen*: It's true!*
Teacher*: There, I'd like to understand what you have seen, raise your hands. Observe them carefully, reflect and come in.*

The teacher sets up the contract. She knows that the pupils are astonished and exploit this "surprise" to contextualize in the classroom the difficult metacognitive question of the Chinese textbook. In the discussion low achievers explain their difficulties and ask for other sets of problems to understand better, while a lot of interesting ideas are shared.

There are always the same numbers (at least in the first two rows).
There are always the same operations (additions, subtractions).
Subtractions are "additions with dots": "... + 45 = 75".
In the first column only additions; then only subtractions.
Every problem has the solution in another problem.

The conclusion is clearly stated by Arianna: *You had to understand that everything was linked. Also the last ones, that were different, were linked to the same story of ducks. The numbers were always the same.*

4.2. The second task (February 3, 2010)

Task. 21 pupils are present and are split into pairs (2 pupils work together with Giacomo to help him). Each pair is given the following task: *Invent three problems like the Chinese ones of the first row*.

Results. 3 pairs invent problems correctly without any help; 4 pairs receive little help from the teacher; 2 pairs and the trio have to work more, with more help from the teacher (e.g. at the beginning they do not maintain the same context in the three problems). All the pupils (but one pair) make examples about animals with a very little shift from the original problems. Most pupils introduce narrative elements.

Mathematical Discussion. Some days later the teacher asks the pupils to read aloud their problems and invites all the class to make comments. Some pupils consider the task very easy and others very difficult. A set of problems with very concise text without narrative aspects are criticized because "they are only operations".

4.3. The third task (February 24, 2010)

Task. 19 pupils are present and are split more or less into the same pairs as earlier (with Giacomo supported by two friends). Each pair is given the following task: *This is the second row of the Chinese problems. Try to construct three similar ones, maintaining the same arithmetical typology*. The teacher aim, with this change in the statement, is to try to focus more on the arithmetic structure.

Results. 3 pairs work well autonomously; 2 pairs and the trio need a little help; 3 pairs need more help from the teacher. 5 pairs produce the problems starting from the example problem of the second row.

Mathematical discussion. Immediately after, the pupils are called to read aloud their problems and to comment. They agree that this task is more difficult than the second one and they have started from the example problem (the third), then produced the second and then the first, the most difficult.

4.4. The fourth task (March 24, 2010)

Task. 21 pupils are present. They are given again the table of the nine problems (with a space below each of them) and, on a separate sheet, the nine graphic schemes of the table 1 in a random order. They have to answer individually the following question: *Observe, try to understand what the scheme mean, pair them with the nine problems (cut and paste the right scheme below each problem) and explain why*. As the task is very unusual, before the individual solution, one case is analysed under the teacher's guidance.

Teacher*: Many have said that choosing the scheme for the first problem is quite easy. Why? How have you understood which scheme matches the first problem?*
Sofia*: If you look carefully, you see that the scheme is <u>equal</u> to the text; in both we have 45 white ducks and 30 black ducks and we are asked how many all together and in the scheme there is a question mark.*
Lorenzo*: The Chinese word* [the classifier or measure word 只 zhī] *close to the numbers might mean ducks, we don't know, it is always the same and the text is about ducks.*
Fabio*: In the scheme the parentheses are longer when the number is greater and shorter if the number is smaller. They are not equal: there are long ones, medium ones and short ones.*
Mohassen*: The red parenthesis in the first text means that the result is larger than the two numbers, as it is longer*.
Lorenzo*: If you look, you see that the red parenthesis is the number to be found, as there is always a question mark.*
Arianna*: The two blue parentheses represent the data, whilst the red one is the number to be found. There is a scheme for each problem. Yet there are equal schemes, because there problems that are solved in the same way.*
Teacher*: With this indications/observations try to pair each text with its scheme.*

Good work!

Results. All the pupils solve the task correctly although in different times. In a limited number of cases the teacher helps for the problems of the third row. The strategies used by the pupils are mostly based on the length and the colour of parentheses that orient them towards algebraic reasoning (rather than on arithmetic reasoning). Most pupils become aware that if a+b=c then c-a=b and c-b=a, independently from the specific value of the numbers a,b,c.

4.5. Long term effects.

The pupils do not work on Chinese problems any more. At the beginning of the third grade (September), the teacher gives the following task about multiplicative structures, following examples in other Chinese textbooks: *Read carefully the three following texts, make the necessary operations and answer the questions. What can we say about these problems?*

Max is training for a footrace. Every day he runs 3 km. He trains for 8 days. How many km does he run?	Max is training for a footrace. Every day he runs 3 km. He has run 24 km. How many days has he run?	Max is training for a footrace. In 8 days he runs 24 km. Every day he runs the same distance. How many km does he run every day?

Table 2. Training for a footrace

They immediately guess that they are "Chinese problems" and look for the solutions examining the *biànshì* problems. In other words, Chinese problems force a new strategy, by analogy (Polya, 1954). This is only a hint that has to be interpreted later. The pupils of this classroom have not been trained much about this kind of problems. Nevertheless they seem to have acquired some special attitude (for instance to link addition and subtraction), that makes them better problem solvers than second graders in other classrooms. Indirect evidence shows at the end of the second grade, when the national assessment in Italian and Mathematics is made by INVALSI[3]. One of the most difficult items at the national level is the following (requiring the combination of multiplication and subtraction): *The teacher has 3 boxes of 8 pencils each and gives a pencil to each of her 22 pupils.* Three answers are given: *A) No pencil is left; B) One pencil is left; C) Two pencils are left.* In this classroom the percentage of good solutions was higher than the national one.

5. EXAMPLE 2: MULTIPLICATIVE PROBLEMS IN THE FOURTH GRADE (Rita Canalini)

Context. This fourth grade classroom (23 pupils) has been taught by the same teacher (Rita Canalini) from the first grade on: 5 pupils are of not Italian extraction, but are fluent in Italian; one pupil is mentally retarded with a special

[3] The Italian National Institute for School Assessment (http://www.invalsi.it/invalsi/index.php).

syllabus. The pupils come from low social rank families with only two parents with a university degree. Pupils are accustomed to describe in writing the strategies and the problems met and to produce texts about mathematical activity. The teacher often resorts to *written dialogue*. They have already solved (in the third grade) the nine problems on ducks. The teacher introduces now multiplicative structures.

5.1. The first task (September 22 - October 4, 2010)

Task. 2 pupils are absent: 21 pupils are split into pairs (and a trio), with a high or medium level pupil in each group. Each group is given the problems in the table 3 below, with the following tasks: *1) Read very carefully the texts and complete them with the missing questions and headings; 2) Solve every problem in the space below with a graphic scheme. As the space is limited, use signs, words, and numbers. Then write the operation. 3) Why these problems have been given together? What was the most difficult? Why?*

A)…………………….. Teacher Anna Maria has marked 6 admission tests for each of her 23 pupils. How many tests has she marked?	B)............................... Teacher Anna Maria has marked 138 tests. Each of her pupils has taken 6.	C)............................... Teacher Anna Maria has marked 138 tests. Tests have been taken by her 23 pupils. ....

Table 3: Teacher Anna Maria and the admission tests

Results. The missing questions of the problems are consistently stated by 7 groups among 10. Half of the groups (5) produce schemes (see an example in the fig. 2). The scheme is the same suggested by Vergnaud (1988) for the conceptual field of multiplicative structure. The teacher recognizes it and decides to exploit it in further activity.

Mathematical Discussion. About two weeks later, according to the didactical cycle, pupils are engaged in a mathematical discussion orchestrated by the teacher. An excerpt follows:

Teacher: [about the problem B, the most intriguing one] *Some pupils think that it is strange to start from two numbers that count tests and to get a third number that counts children. 138 count tests, 6 counts tests. How is it possible that an operation between these numbers produces a number that counts children? [...]*

Figure 2:
Samantha and Alyssa's scheme

Samantha [the author with Alyssa of the scheme in the fig. 2]*: Among those problems, it tells that each child makes 6 tests hence 1 comes out, that is [the child] who makes the tests.*

Teacher*: If I have understood well, Samantha says that it is not true that in the text B there are only numbers counting tests. She*

sees also a 1 that counts the child who makes 6 tests.
Basma*: That 1 counts each child making 6 tests. [...]*
Donato*: Samantha and Alyssa are right. They make one understand that tests are 138, pupils are 23, and tests made by one pupil are 6.*
Gabriele*: Good boy! Donato is good because he has explained that 6 is the tests made by the pupils, 138 is the tests collected from all pupils and 1 is the child who makes the tests.*

This short excerpt starts the discussion about a crucial issue: without introducing 1 as a fourth term, it is not possible to understand why it is possible to get a number of pupils from two numbers counting tests. High achievers succeed in taking part consistently in the discussion. When the teacher mirrors, also low achievers intervene. Their interventions are not always at the same level of the others, but, nevertheless, this exposure to high level discussion will be useful in the following. The teacher asks all the pupils to copy the scheme of the fig. 2 and to highlight the links and the operations that connect the four numbers. This is a way to involve all the pupils, leaving open the possibility that each contributes at his/her own level.

5.2. The second task (October 18, 2010)

Task. 22 pupils are present. Each pupil is given the problems in the table 4 below, with the following tasks: *1) A child has solved the three problems building the schemes below. Cut and match each scheme to one problem, then solve it by an operation. 2) Why has the child decided to use these schemes? What do you think?*

Mario sticks his photos in an album. He fills 9 pages and sticks 6 photos in each page. Calculate how many photos Mario has stuck.	Mario fills 9 pages of his album to stick 54 photos. Each page features the same number of photos. How many photos are stuck in each page?	Mario sticks 54 photos in his album. He sticks 6 photos in each page. How many pages has he used?

P	F
1	?
9	54

P	F
1	6
?	54

P	F
1	6
?	54

Table 4. Mario and the photo album (P means pages; F means photos)

Results. All the pupils but 3 answer the question 1 correctly. Most produce high level texts to answer the question 2. Some excerpts follow:

The child has used these schemes to help himself to understand better the numbers. The table is divided into two parts: in one part it shows the pages used by Mario and in the other it shows the photos stuck by Mario.
The scheme shows the unknown (sic!) *and sums up the problem. It helps to find*

the operation.
[...]Without P and F one cannot understand the number of the object you mean.
[In the text] there is not a fourth number, that is always in the text and in the schemes; this number is "1" that I have encircled in the schemes and in the text. Yet in the text there is not a number but a word hence I have encircled the word "each".

The last statement is produced by Giulia who shows to master well the subtleties of Vergnaud's scheme. The teacher gives her another task, by written dialogue: *Maria has 12 eggs in 2 containers. How many eggs would she have if she had 8 containers equal to the ones she already has?* Giulia solves the problem enlarging the scheme and inventing the reduction to unit.

Figure 3:
Giulia's new scheme

In short, the pupils find different functions in the scheme: it clearly identifies two "measure" spaces; it identifies the unknown; it helps to identify the operation; it helps to master the complex issue of measuring units. The invention and the quick diffusion of the scheme suggested by Vergnaud (1988) for multiplicative structures is a high level performance: it might be understood knowing the history of the classroom, taught from the first grade to produce drawings, schemes and symbols in problem solving.

6. DISCUSSION

The two teaching experiments show how two expert teacher-researchers have exploited the CAC prompted by the Chinese method of *biànshì* problems together with elements of the western research tradition, e.g. the didactical cycle of the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) and the scheme for the conceptual field of multiplicative structure (Vergnaud, 1988). As teacher-researchers they have been ready to take up challenges and to collect detailed data from the classroom, yet the decision of testing this kind of problems has been slow and well thought-out. For them too, the experiments have required and provoked some restructuring of beliefs about mathematics teaching and learning. They were already familiar with research on word problems and knew the semantic analysis of additive word problems (Carpenter et. al, 1982) and the analysis of the conceptual field of multiplicative structures (Vergnaud, 1988). Yet, the Chinese method of *biànshì* problems opened a new perspective, fostering the recourse to system of problems (rather than to separate ones). The systemic presentation of problems contrasts the attitude of teachers' guides in the West, which aims, rather, to classify, separate and order (according to the difficulty levels) word problems. *Fragmentation and connectedness* are identified by Ma (1999) as two opposite features that characterize (among others) teaching attitudes in US and China in many areas of arithmetic.

Connectedness applies also to this case of arithmetic word problems: it is the very presentation of these problems as connected to each other to make possible the understanding and the invention of general solving schemes. This approach, prompted by the Chinese case but tailored to the western research tradition, has paved the way to *authentic algebraic reasoning* for very young learners. Not only the CAC but also the observation of the processes in the teaching and learning activity has restructured some beliefs of the teacher-researchers about mathematics teaching and learning and focused on open problems.

The teaching experiments have raised some research questions. For instance, we have discussed together the linguistic structure of word problems. The nine problems of ducks have a very dry and concise structure. We have tried to maintain it in the translation. It mirrors the structure of the Chinese language and meets the need to avoid the introduction of too many characters in the second grade, when the mastery of writing is still very limited for Chinese pupils. The problems in the table 1 have a very repetitive structure where data and unknown exchange their places from one problem to another. This repetition seems to foster the match between problems and schemes. According to the western tradition, paradigmatic statements are supposed to be demanding as they require narrative interpretation (Bruner, 1986). For instance, word problems have a lot of narrative enrichment, that is expected to make them closer to children's experience (for instance, one could read that it was a sunny day and the ducks were happy to swim in the sun). Does a richer narrative text help or inhibit the arithmetic/algebraic solution? From a cognitive point of view, is the interpretation of a narrative text more demanding or more helpful compared to a paradigmatic text? Does it apply in the same way to high and low achievers, to pupils of Italian extraction and to pupils coming from other cultures?

The reports of the teaching experiments by Ferri and Canalini (and others) have enriched the discussion in the last workshop (October 2010 – May 2011). The process of defying participants' beliefs has been started. Only about one third of the 26 registered participants (results collected on May 24th, 2011) have changed the practice in their own classroom (involving also a number of colleagues in their schools). Actually the impact of CAC on teachers' practices is not analyzed by Boero & Guala (2008), who mainly cope with tasks for pre-service teacher education. When practicing teachers are introduced to CAC, one observes different phases: 1) to query one's own beliefs; 2) to take part in the design of innovative classroom activities where CAC is exploited; 3) to test the designed activities in one's own classrooms. The shift from one phase to another is slow and fragile, especially the last one concerning the translation of the CAC into practice: there is resistance to change classroom activities even when they have shown unsuccessful. Is it possible (and how) to overcome this resistance?

REFERENCES

Bartolini Bussi M. G. (2011), Artefacts and utilization schemes in mathematics teacher education: place value in early childhood education, Journal for Research on Mathematics Teacher Education, vol. 14, 93-112.

Bartolini Bussi M. G., Garuti R., Martignone F. & Maschietto M. (2011), Tasks for teacher education in the MMLAB-ER project. proc. PME 35.

Bartolini Bussi, M. G. & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English & al. (Eds.), Handbook of International Research in Math. Educ., second edition (pp. 746-783). Routledge.

Boero P., Guala E. (2008). Development of mathematical knowledge and beliefs of teachers. In Sullivan, P. & Wood, T. (Eds.), The International Handbook of Mathematics Teacher Education. (Vol. 1, pp. 223-244), Purdue University, USA: SensePublishers.

Bruner J. (1986), Actual minds, possible worlds, Cambridge University Press.

Carpenter T. P., Moser J.M. & Romberg T. A. (Eds.) (1982), Addition and subtraction: A cognitive perspective. Hillsdale, NJ: Erlbaum.

Cai J. & Nie B. (2007), Problem solving in Chinese mathematics education: research and practice, ZDM the International journal of Mathematics Education, 39:459–473.

Gu L., Huang R., Marton F. (2004), Teaching with variation: A Chinese way of promoting effective mathematics learning, in L. Fan, N.Y. Wong, J. Cai & S. Li (eds.), How Chinese learn mathematics: perspective from insiders, 309-347, Singapore. World scientific.

Ma L. (1999). Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States, Lawrence Erlbaum.

Mariotti M. A. & Maracci M. (in press) Resources for the teacher from a semiotic mediation perspective. In G. Gueudet, B. Pepin, L. Trouche, From Text to Lived Resources: Mathematics Curriculum Materials and Teacher Development, Springer.

Polya, G. (1954). Mathematics and Plausible Reasoning: Induction and Analogy in Mathematics (Vol.1) PrincetonUniversity Press.

Shu Xue (1996), ISBN 7-200-02567-4

Spagnolo F. & Di Paola B. (eds.) (2010), European and Chinese cognitive styles and their impact on teaching mathematics, Berlin: Springer.

Sun, X. H., Wong, N. Y., & Lam, C. C. (2005). Bianshi problem as the bridge from “entering the way” to “transcending the way. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education*, 9(2), 153-172.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.) Number Concepts and Operations in the Middle Grades (pp. 141-161). Reston: NCTM.